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REPRESENTATION THEOREMS AND
INEQUALITIES FOR HERMITIAN MATRICES

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Summary

In a series of papers, it has been shown that the integral representation

$$(1) \qquad \frac{\pi^{N/2}}{|A|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,Ax)} dx$$

where A is a symmetric matrix whose real part is positive definite, $dx = dx_1 dx_2 \ldots dx_N$, and the integration is over the whole x-space, can be used to establish a number of inequalities derived in other ways.

On the other hand, the methods used by other authors, Ostrowski and Taussky, Ky Fan, and Oppenheim, possess the advantage of being equally applicable to the study of hermitian matrices, whereas (1), as it stands, can be used only for symmetric matrices.

In this paper we shall establish an analogue of (1) for positive definite hermitian matrices, and then use this result to derive a number of known inequalities.

As a further example of the use of representation theorems of this type we shall use (1) and the hermitian analogue to derive a partial generalization of a recent inequality of Hua. We shall then use a deeper representation theorem of Siegel and Ingham to obtain a further generalization. The result for hermitian matrices requires a generalization of the Siegel result due to Braun. A generalization in a different direction enables still further results to be obtained.

In a recent paper by Marcus, extensions of a still different nature are indicated.

REPRESENTATION THEOREMS AND INEQUALITIES FOR HERMITIAN MATRICES

Richard Bellman

1. Introduction

In a series of papers, [1], [3], [4], it has been shown that the integral representation

$$(1) \qquad \frac{\pi^{N/2}}{|A|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,Ax)} dx$$

where A is a symmetric matrix whose real part is positive definite, $dx = dx_1 dx_2 \ldots dx_N$, and the integration is over the whole x-space, can be used to establish a number of inequalities derived in other ways; cf. Ostrowski and Taussky, [11], Ky Fan, [8], Oppenheim, [10].

On the other hand, the methods used by the authors of the papers cited above possess the advantage of being equally applicable to the study of hermitian matrices, whereas (1), as it stands, can be used only for symmetric matrices.

In this paper we shall establish an analogue of (1) for positive definite hermitian matrices, and then use this result to derive a number of known inequalities.

As a further example of the use of representation theorems of this type we shall use (1) and the hermitian analogue to derive a partial generalization of a recent inequality of Hua, [6]. We shall then use a deeper representation theorem of Siegel, [12], and Ingham, [7], to obtain a further generalization.

The result for hermitian matrices requires a generalization of the Siegel result due to Braun, [5]. A generalization in a different direction, given in [2], enables still further results to be obtained.

In a recent paper by Marcus, [9], extensions of a still different nature are indicated.

2. Evaluation of an Integral

By analogy with the formula of (1.1), let us discuss the integral

$$(1) \qquad J(H) = \int_{-\infty}^{\infty} e^{-(\bar{z},Hz)}dxdy,$$

where $z = x + iy$ and H is a positive definite hermitian matrix.

Write $H = A + iB$, A and B real, so that $A' = A$, $B' = - B$. It follows that

$$(2) \qquad J(H) = \int_{-\infty}^{\infty} e^{-(x,Ax)-2(Bx,y)-(y,Ay)}dxdy.$$

Since the integral is absolutely convergent, it may be evaluated by integration first over x and then over y.

Using the relation, (recall that $B' = - B$),

$$(3) \qquad (x,Ax) + 2(Bx,y) = (x,Ax) - 2(x,By)$$

$$= (A(x - A^{-1}By),x - A^{-1}By) - (By,A^{-1}By),$$

we see that

(4) $$\int_{-\infty}^{\infty} e^{-(x,Ax)-2(Bx,y)}dx = \frac{\pi^{N/2}}{|A|^{1/2}} e^{(By,A^{-1}By)}.$$

Hence

(5) $$J(H) = \frac{\pi^{N/2}}{|A|^{1/2}} \int_{-\infty}^{\infty} e^{-[(y,Ay)+(y,BA^{-1}By)]}dy$$

$$= \frac{\pi^{N/2}}{|A|^{1/2}} \cdot \frac{\pi^{N/2}}{|A + BA^{-1}B|^{1/2}}$$

$$= \frac{\pi^{N}}{|A|\ |I + A^{-1}BA^{-1}B|^{1/2}}.$$

3. Relation between $J(H)$ and $|H|$

It remains to express $J(H)$ in terms of $|H|$. We have

(1) $$|H| = |A + iB| = |A|\ |I + iA^{-1}B|$$

$$|H'| = |A - iB| = |A|\ |I - iA^{-1}B|.$$

Since $|H| = |H'|$, we have

(2) $$|H|^2 = |A|^2\ |(I + iA^{-1}B)(I - iA^{-1}B)|$$

$$= |A|^2\ |I + A^{-1}BA^{-1}B|.$$

Combining the foregoing result with (2.5), we obtain the desired result.

Theorem 1.

(3) $$J(H) = \pi^N/|H|.$$

4. Extensions

It is easy to see, following, for example, the argument

given in [4], that (3.3) is valid for a matrix of the form $H_1 + iH_2$, provided that H_1 is a positive definite hermitian matrix, and H_2 is hermitian.

5. An Inequality of Ostrowski and Taussky

If $H = H_1 + iH_2$, where H_1 and H_2 are hermitian, we have

$$(1) \qquad J(H_1 + iH_2) = \int_{-\infty}^{\infty} e^{-(\bar{z},H_1 z)} e^{-i(\bar{z},H_2 z)} dxdy,$$

with $(\bar{z},H_1 z)$ and $(\bar{z},H_2 z)$ both real. Hence,

$$(2) \qquad J(H_1 + iH_2) \leq J(H_1),$$

with strict inequality unless $H_2 = 0$.

This yields the inequality

$$(3) \qquad |H_1 + iH_2| \geq |H_1|,$$

whenever H_1 is positive definite, with strict inequality unless $H_2 = 0$. By continuity, we see that the inequality remains valid for H_1 non-negative definite. This is a result due to Ostrowski and Taussky, [11].

As pointed out in [1], an immediate consequence of (3) is the inequality

$$(4) \qquad |A + B| \geq |A|,$$

whenever A is a real symmetric matrix which is non-negative definite, and B is skew-symmetric. We shall use this result below.

Closely related to the foregoing result is the inequality

(5) $$|I - RU| \geq |I - R|,$$

where R is a non-negative definite hermitian matrix with $I - R$ non-negative definite and U unitary.

For closely related and further results, see Taussky, [13].

6. Some Determinantal Inequalities

Using Theorem 1 and Hölder's inequality, we readily derive the inequality

(1) $$|\lambda H_1 + (1-\lambda)H_2| \geq |H_1|^{\lambda}|H_2|^{(1-\lambda)},$$

provided that H_1 and H_2 are positive definite hermitian and $0 \leq \lambda \leq 1$; cf. Oppenheim, [10], and the corresponding result for symmetric matrices in [1].

7. Hua's Inequality for Symmetric Matrices

In a recent paper, Hua, [6], established by means of representation theorems of a different type, the following result:

Theorem 2. *Let $X_1, X_2, \ldots, X_m$ be complex matrices of order N with the property that $I - X_i^* X_i$ is positive definite hermitian for $i = 1, 2, \ldots, m$. Then for $\rho > 0$*

(1) $$(|I - X_i^* X_j|^{-N-\rho+1})$$

is non-negative definite.

As we shall show, the exponent $-N-\rho+1$ can be

considerably improved. One method will yield an improvement in one direction, while the other method will yield an improvement of different type.

Let us begin by establishing

<u>Theorem 3</u>. *If* $A_1, A_2, \ldots,$ *is a set of real matrices of order* N, *then*

$$(2) \qquad (|I - A_i'A_j|^{-k/2}), \quad i,j = 1,2,\ldots,M,$$

is positive definite for $k = 1,2,\ldots,$ *provided that* $I - A_i'A_i$ *is positive definite for* $i = 1,2,\ldots,M$.

<u>Proof</u>. Let us begin with the observation that every real matrix B can be written in the form $B = B_s + B_a$ where B_s is symmetric and B_a is skew-symmetric. Thus

$$(3) \qquad B_s = (B + B')/2,$$

$$B_a = (B - B')/2.$$

Observe also that

$$(4) \qquad (x, Bx) = (x, B_s x).$$

We also wish to use the fact that

$$(5) \qquad A_i'A_i + A_j'A_j - (A_i'A_j + A_iA_j') = (A_i' - A_j')(A_i + A_j)$$

is non-negative definite. It follows from our assumptions concerning $I - A_i'A_i$ that $I - (A_i'A_j)_s$ is positive definite.

Referring to §4, we can then conclude that

(6) $$|I - A_i'A_j| \geq |I - (A_i'A_j)_s|.$$

It follows that Theorem 3 will be demonstrated if we prove that

(7) $$(|I - (A_i'A_j)_s|^{-k/2})$$

is positive definite.

Now

(8) $$\frac{\pi^{N/2}}{|I - (A_i'A_j)_s|^{1/2}} = \int_{-\infty}^{\infty} e^{-(x,(I-A_i'A_j)x)} dx$$
$$= \int_{-\infty}^{\infty} e^{-(x,x)} e^{(A_ix,A_jx)} dx.$$

To show that $(e^{(A_ix,A_jx)})$, $i,j = 1,2,\ldots,N$ is non-negative definite for all x, we begin by showing that $((A_ix,A_jx))$ is non-negative definite. This follows readily from the fact that

(9) $$\sum_{i,j=1}^{N} u_iu_j(A_ix,A_jx) = \left(\sum_{i=1}^{N} u_iA_ix, \sum_{i=1}^{N} u_iA_ix\right).$$

To continue, we require a useful result of I. Schur,

Lemma. If $A = (a_{ij})$ and $B = (b_{ij})$ are both non-negative definite, then $(a_{ij}b_{ij})$ is non-negative definite.

It follows that $((A_ix,A_jx)^k)$, $k = 1,2,\ldots,$ is non-negative definite, and hence that $(e^{(A_ix,A_jx)})$ is non-negative definite. Since $(e^{(A_ix,A_jx)})$ can be non-negative definite on a set of N-dimensional measure zero, at most, it follows that $(|I - (A_i'A_j)_s|^{-1/2})$ is actually positive definite.

To obtain the result stated in Theorem 3, we apply the foregoing lemma to this last result.

8. Hermitian Matrices

In exactly the same fashion, we can use the representation of Theorem 1 to establish

<u>Theorem</u> 4. <u>If</u> $I - H_i^* H_i$ <u>is positive definite hermitian for</u> $i = 1,2,\ldots,$ <u>then</u>

$$(1) \qquad (|I - H_i^* H_j|^{-k}), \quad i = 1,2,\ldots,M,$$

<u>is positive definite for</u> $k = 1,2,\ldots$.

This is the first extension mentioned above.

9. Ingham-Siegel Representation

In order to establish a result for a less discrete set of exponents, we shall employ a more recondite identity, discovered independently in equivalent forms by Ingham, [7], and Siegel, [12].

Here we shall restrict ourselves to the real version. The hermitian analogue may be found in Braun, [5]. The identity is

$$(1) \qquad \int_{X>0} e^{-tr(XY)} |X|^{s-(N+1)/2} dV = c_N(s)/|Y|^s,$$

where

$$(2) \qquad c_N(s) = \pi^{N(N-1)/4} \Gamma(s)\Gamma(s - 1/2)\cdots\Gamma(s - (N-1)/2).$$

The real part of s is assumed to be greater than $(N-1)/2$.

Here X and Y are symmetric matrices of order N, $dV = \prod_{i \le j} dx_{ij}$, and the integration is over the region in

x_{ij}-space where X is positive definite.

The proof proceeds along the same lines as before once we observe that

$$(3) \qquad tr(XY) = tr(XY_s),$$

provided that X is positive definite. This follows from

$$(4) \qquad tr(XY) = tr(X^{1/2}YX^{1/2}) = tr(X^{1/2}Y_sX^{1/2}) + tr(X^{1/2}Y_aX^{1/2})$$

$$= tr(X^{1/2}Y_sX^{1/2}) = tr(XY_s).$$

Furthermore,

$$(5) \qquad (tr(XA_i'A_j)), \quad i,j = 1,2,\ldots,M$$

is non-negative definite if X is positive definite.

The final result is

<u>Theorem</u> 5. <u>Under the assumptions of Theorem</u> 3, <u>the matrix</u> $(|I - A_i'A_j|^k)$ <u>is positive definite provided that</u> $k > (N-1)/2$.

What the precise exponent should be is not clear.

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